Indian Statistical Institute, Bangalore Centre
M.Math I Year, Second Semester
Mid-Sem Examination - 2013-2014
Functional Analysis

Time: 3 Hours   March 5, 2014   Instructor: T.S.S.R.K. Rao

8*5=40

Answer all the question. Give complete answers.

1. Show that the vector space of all polynomials on $[0,1]$, is not a Banach space under any norm.

2. Let H be a complex Hilbert space. Let $x_1 \cdots x_k$ be a set of orthonormal vectors. For any $x \in H$, and complex numbers $z_1, \cdots z_k$, show that

$$|| \ x - \sum_{j=1}^{k} \ z_j \ x_j \ || \geq || \ x - \sum_{j=1}^{k} < x, x_j > x_j \ ||$$

3. Let X be a normed linear space and let $T : X \longrightarrow X$ be a linear map such that $x_n \longrightarrow 0$ implies, $T(x_n)_{n\geq 1}$ is bounded. Show that T is continuous.

4. Let $\{X_n\}_{n\geq 1}$ be a sequence of Banach spaces. Let $Y = \{\{x_n\}_{n\geq 1} : x_n \in X_n$ and $\lim\limits_{n\longrightarrow\infty} || \ x_n \ || = 0\}$. Show that Y is a Banach space.

5. Let Y be as in question 4. Let $x_n^* \in X_n^*$ be a sequence such that $\sum\limits_{n=1}^{\infty} \ || \ x_n^* \ || \ < \infty$. Define $T : Y \longrightarrow \mathbb{R}$ by $T(\{x_n\}_{n\geq 1}) = \sum\limits_{n=1}^{\infty} \ x_n^*(x_n)$. Show that T is a well-defined, linear map. Show that $|| \ T \ || = \sum\limits_{n=1}^{\infty} \ || \ x_n^* \ ||$.

6. Consider the Hilbert space, $\ell^2 = \{\{\alpha_n\}_{n\geq 1} : \sum \ | \ \alpha_n|^2 < \infty\}$. Let $f : \ell^2 \longrightarrow \mathbb{C}$ be a linear map, that is <u>not</u> continuous. Show that $\ker f$ is a dense subspace of ℓ^2. (Hint: Let $B(0,1)$ be the unit ball. Show $f(B(0,1)) = \mathbb{C}$).

7. Let $\{f_n\}_{n\geq 1} \subset L^1[0,1]$ be such that $\int\limits_0^1 \ | \ f_n| \ d\lambda \longrightarrow 0$. Show that $\exists$ a subsequence $\{f_{n_k}\}_{k\geq 1}$ of $\{f_n\}_{n\geq 1}$ such that $f_{n_k} \longrightarrow 0$ a.e.

8. Show that the space of continuous functions with compact support on $\mathbb{R}$ is not a Banach space w.r.t the supremum norm.